

# Lecture Notes 6: General Equilibrium (cont'd)

- Pareto Optimality (taking a closer look)
- Optimality  $\leftrightarrow$  Equilibria: the Welfare Theorems
- Problems: Existence, Stability, Uniqueness
- All those Assumptions

Last week we looked at two specific cases (exchange and Robinson Crusoe) to learn more about competitive equilibria. This time we will revisit the connection between competitive equilibria and Pareto efficiency in slightly more general terms. Subsequently, we will emphasize the potential problems of equilibrium analysis and conclude by listing all those (unrealistic) assumptions behind the perfect model we have developed over the weeks.

# Pareto Optimality

Recall the definition of a Pareto optimum: an allocation is **Pareto optimal** or efficient iff there is no alternative that leaves no-one worse off but makes at least one person better off.

Let us characterize a Pareto optimum in an economy with 2 consumers (A and B), two goods (1 and 2), and two exogenously given factors (K and L):

$$\begin{aligned} \max_{x_1^A, x_2^A, x_1^B, x_2^B, K^1, K^2, L^1, L^2} \quad & U^A(x_1^A, x_2^A) \\ \text{s.t.} \quad & U^B(x_1^B, x_2^B) = \bar{U}^B \\ & x_1^A + x_1^B = F^1(K^1, L^1) \\ & x_2^A + x_2^B = F^2(K^2, L^2) \\ & K^1 + K^2 = \bar{K} \\ & L^1 + L^2 = \bar{L} \end{aligned}$$

Person A is the person we are trying to make better off (we could do the same for B). The first constraint makes sure that no one else will be worse off. The last four constraints make sure that the allocation is feasible or technically possible: We cannot use more factor inputs in production than there are and we cannot consume more output than is produced.

Substituting the last four constraints for  $x_1^A, x_2^A, K^1$ , and  $L^1$  results in an optimization with only one constraint (the first) which can be solved by setting up the Lagrangean:

$$\max_{x_1^B, x_2^B, K^2, L^2, \lambda} U^A(F^1(\bar{K} - K^2, \bar{L} - L^2) - x_1^B, F^2(K^2, L^2) - x_2^B) + \lambda(U^B(x_1^B, x_2^B) - \bar{U}^B)$$

The FOCs are:

$$\begin{aligned} x_1^B : & \quad -U_1^A + \lambda U_1^B = 0 \\ x_2^B : & \quad -U_2^A + \lambda U_2^B = 0 \\ K^2 : & \quad -U_1^A F_K^1 + U_2^A F_K^2 = 0 \\ L^2 : & \quad -U_1^A F_L^1 + U_2^A F_L^2 = 0 \end{aligned}$$

The first two FOCs can be combined (dividing one by the other) to show that  $MRS^A = MRS^B$ . More generally (with many goods and consumers), the MRSs must be equal across consumers:  $MRS = MRS^A = MRS^B = \dots$ . This represents efficiency on the consumption side or **consumption efficiency**.

The third and fourth FOCs can be rewritten as:

$$MRS^A = \frac{F_K^2}{F_K^1} = \frac{F_L^2}{F_L^1}$$

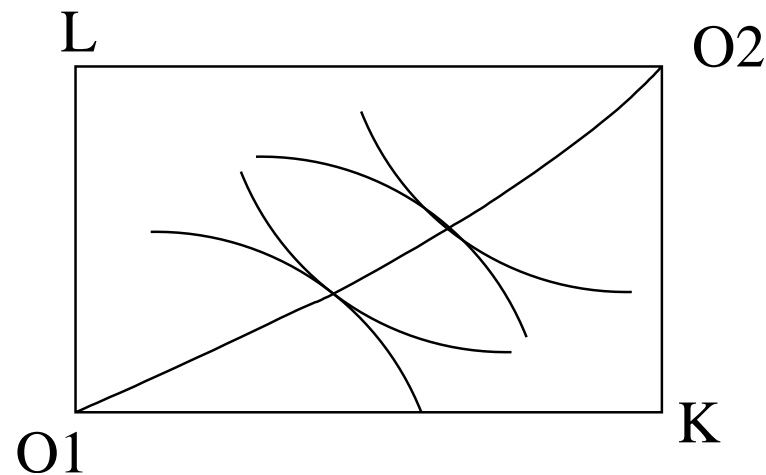
Note that the second and third term are simply the marginal rate of transformation (between outputs 1 and 2). So we see that the MRS (equal across different consumers) must equal the MRT. This is called **overall efficiency**, i.e. efficiency across consumption and production.

Finally, rewrite the second equality above as:

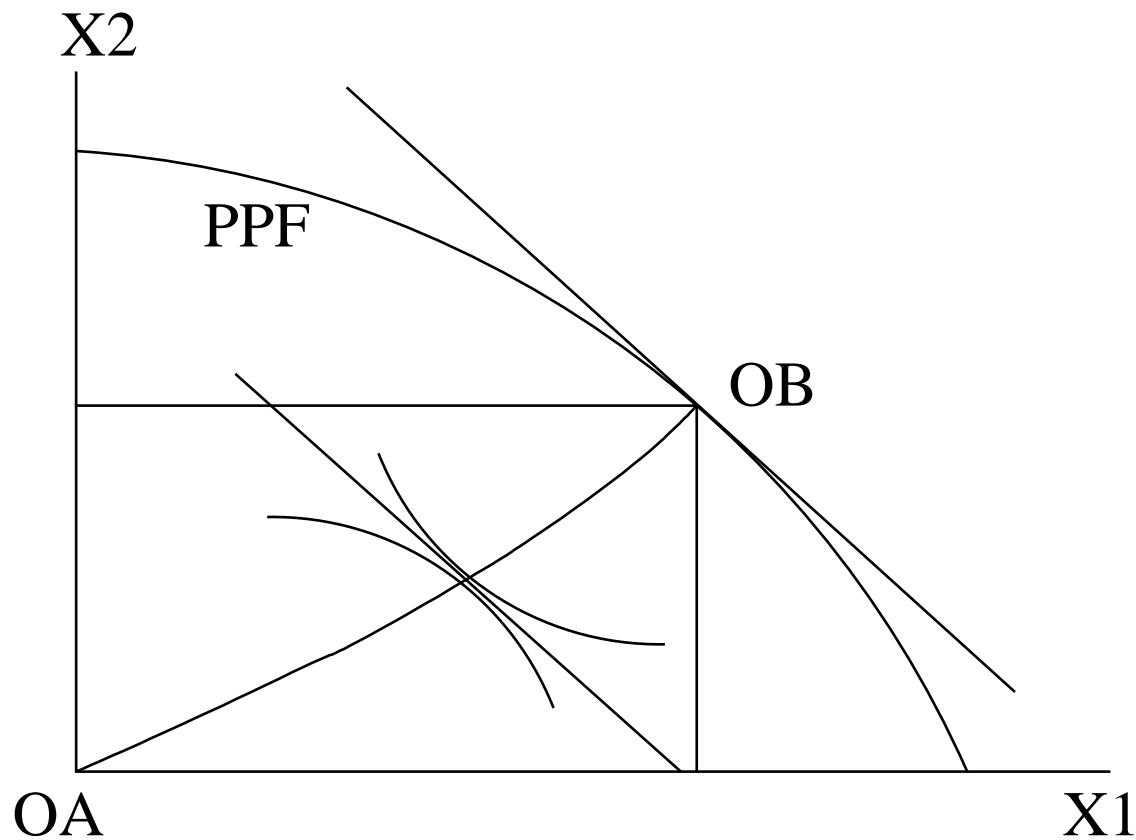
$$MRTS^1 \equiv \frac{F_K^1}{F_L^1} = \frac{F_2^K}{F_2^L} \equiv MRTS^2$$

So we see that on the production side the MRTSs must be equal across sectors. This represents efficiency on the production side or **production efficiency**.

All three efficiency conditions can be represented graphically:



This is an Edgeworth box for the production side. Tangency inside this box represents production efficiency, MRTSs are equal across sectors. Wandering along the contract curve and keeping track of output quantities gives rise to the production possibility frontier (PPF) in the next diagram:



Inside the (consumption) Edgeworth box tangency represents equal MRSs across consumers or consumption efficiency. But the Edgeworth box must be inserted underneath the PPF in such a way that the latter's slope (the MRT) at that point equals the MRS inside the box so that  $MRS = MRT$  or overall efficiency.

To see why all these equalities ensure optimality, suppose there was an inequality. In the case of (no) consumption efficiency, there would then be room for a mutual increase in respective utilities. Similarly, if production efficiency did not hold, factors could be reallocated to achieve higher output(s). In the case of over-all efficiency, resources could be shifted out of one sector into another and the new output combination would give rise to higher utility on the consumption side.

# Connection between Market Equilibria and Pareto Efficiency

Intuitively, the connection between competitive equilibria and Pareto efficiency is straightforward. The equalities we identified as consumption, production, and over-all efficiency are usually achieved by the market.

Take the case of consumption efficiency. Our example of an exchange economy showed that since utility maximizing consumers equate their MRSs to relative prices and in a market all face the same prices, the market equates their MRSs by means of the equilibrium price vector.

The same argument holds in factor markets. Profit maximizing firms equate their MRTS to relative factor prices and since all of them face the same relative prices the market achieves production efficiency.

And as Robinson Crusoe illustrated, the same holds for over-all efficiency.



But is it really the case that the set of Pareto optima and the set of possible market equilibria (under different endowments) exactly coincide?

Let us divide this question into two: is every market equilibrium automatically Pareto efficient? And secondly, can every Pareto optimum be achieved or decentralized as a competitive equilibrium?

The answer to the first question is a rather general yes. The first welfare theorem tells us, that every market equilibrium is Pareto efficient. See Jehle & Reny (pp. 218–9) for a simple proof under rather general conditions (non-satiation is required).

The answer to the second is a more limited yes. The second welfare theorem maintains that every Pareto optimum can be achieved as a market equilibrium, but under stricter conditions and subject to the qualification that endowments have to be (re)allocated appropriately. We already discussed the latter point. The stricter conditions are needed to guarantee existence of the competitive equilibrium. This is one of the issues we turn to next.

# Existence, Stability, Uniqueness

When applying general equilibrium analysis and conducting comparative statics, the implicit (unspoken) assumption is that the equilibrium exists, is unique (not always — see below) and stable. What conditions are needed to ensure that this is indeed the case:

- **Existence**

The possible lack of an equilibrium is clearly troubling. In that case equilibrium analysis would be completely meaningless. Now, it is possible to prove existence mathematically. On the consumption side, we require aggregate excess demands (don't even need fcts) to be continuous, homogenous, and satisfy Walras' law. On the production side, production possibilities should be convex. One can then apply so-called fixed point theorems to show existence. Jehle & Reny (pp. 188–97) do this for an easy case where demands are functions and there is no production.

Note that failure to prove existence (under more general/realistic conditions) does not imply that equilibrium cannot exist, it merely might not.

- **Uniqueness**

After worrying about the absence of equilibria, let us consider the possibility that there are too many of them. Regarding uniqueness, one can distinguish local and global uniqueness.

Local uniqueness simply excludes the possibility of having a range of (infinitely) many equilibria in one spot. This possibility is usually an isolated exception (although we already encountered an example where it was not). Once we have local uniqueness, we can show that there is an uneven number of (regular) equilibria, say  $2n + 1$ , of which  $n + 1$  are stable while the other  $n$  are not.

To ensure global uniqueness requires even stronger assumptions. There are two approaches: one involves strong assumptions on the substitutability of goods, the other relies on obscure mathematical conditions.

Note that some theories actually rely on the multiplicity of equilibria.

- **Stability**

Stability of equilibrium corresponds to whether the markets and its anonymous participants will actually find a particular equilibrium. Once we assume global uniqueness stability almost follows. We have one equilibrium, i.e.  $2n + 1 = 1$  or  $n = 0$ , and one ( $n + 1 = 1$ ) is stable. But, of course, to start with global uniqueness implies restrictive assumptions.

The above list/discussion was meant as a brief overview. Rigorous general equilibrium theory is very mathematical.

## Assumptions behind our “perfect” model

Over the first half of the semester we have built a model that relies on numerous assumptions. We have pointed out many but not all of them. Let us try to give a more complete list here. Note that most of these assumptions — taken separately — are not terribly realistic. Nevertheless, we have used them to build a “perfect” model, perfect in the sense of assuming a perfect world. The resulting model is the workhorse or starting point of most of economics. Most subjects studied in economics are approached by relaxing one (or sometimes several at once) of the below assumptions. Therefore the list below also includes which field/subject of economics you arrive at when relaxing that particular assumption:

- closed economy ( $\Rightarrow$  international economics)
- small price-taking actors w/o market power ( $\Rightarrow$  imperfect competition, industrial organization)
- no money nor other financial assets ( $\Rightarrow$  finance)

- no transaction costs, no institutions ( $\Rightarrow$  institutional economics)
- no (true) time dimension ( $\Rightarrow$  macro)
- perfect information ( $\Rightarrow$  informational economics)
- no externalities ( $\Rightarrow$  environmental economics)
- no government, no public goods ( $\Rightarrow$  public finance, political economy)
- ...