Lecture 2: Consumer Theory

- Preferences and Utility
- Utility Maximization (the primal problem)
- Expenditure Minimization (the dual)

First we explore how consumers' preferences give rise to a utility function which describes people's objectives. We then consider two alternative ways of attaining the consumer's optimum. Either she maximizes utility subject to the budget constraint or she minimizes expenditure subject to attaining a given utility level. The second approach (the dual to the first) is less intuitive but much more convenient when deriving further results. It is also very similar to companies' cost minimization.

Utility

Consider two bundles of commodities, i.e., combinations of different goods.
Say bundle A contains x_1^A units of good one, x_2^A units of good two, and so on while bundle B consists of x_1^B units of the first commodity, x_2^B of the second etc.
In short $x^A = (x_1^A, x_2^A, ...)$ and $x^B = (x_1^B, x_2^B, ...)$.

Let there be a binary ordering of any two bundles A and B:

- $x^A \succeq x^B$ means the consumer weakly (including indifference) prefers bundle A over bundle B.
- $x^A \sim x^B$ means the consumer is indifferent (equally likes) A and B.
- $x^A \succ x^B$ means the consumer strictly prefers A over B.

Let us postulate several **axioms** regarding this binary ordering:

- Completeness: consumer knows to compare A and B.
- Reflexivity: $x^A \succeq x^A$ and $x^B \succeq x^B$ (well, technical).
- Transitivity: $x^A \succ x^B$ and $x^B \succ x^C$ implies $x^A \succ x^C$.

These three ensure a complete ordering of all imaginable consumption bundles.

- Continuity (not lexicographic, for example)

These four guarantee existence of a utility function, $U(x_1, x_2, ...)$, describing the preference ordering we started with. It's actually a family of fcts because any monotone transformation (such as adding a constant or taking the natural log) still expresses the same preference ordering.

- Non-satiation: always want more.

This one ensures that there are indifference *curves* and not wide bands of indifference.

- Convexity (of the weakly preferred sets)

Leads to a quasi-concave utility function which is needed as the second order condition of utility maximization (i.e., to make sure it's a maximum and not a minimum).

- Differentiability of the utility function

Pure convenience so we can use calculus.

Utility Maximization

We think of consumers as pursuing individual happiness (formally: maximizing their utility fct) constrained only by what they can afford to buy, the budget constraint.

As for companies and much more realistically here, we assume that the individual has no market power and takes prices as given.

the primal problem:

$$\max_{x_1,x_2,...} \quad U(x_1, x_2, ...) \qquad \text{(the objective)}$$

subject to:

$$I = p_1x_1 + p_2x_2 + ... \qquad \text{(the budget constraint)}$$

We solve this constrained optimization problem by setting up the Lagrangean:

$$\mathcal{L}(x_1, x_2, ..., \lambda) = U(x_1, x_2, ...) + \lambda(I - p_1x_1 + p_2x_2 + ...)$$

FOCs:

$$\frac{\delta \mathcal{L}}{\delta x_1} = U_{x_1} - \lambda p_1 = 0$$
$$\frac{\delta \mathcal{L}}{\delta x_2} = U_{x_2} - \lambda p - 2 = 0$$

dividing one by the other gives:

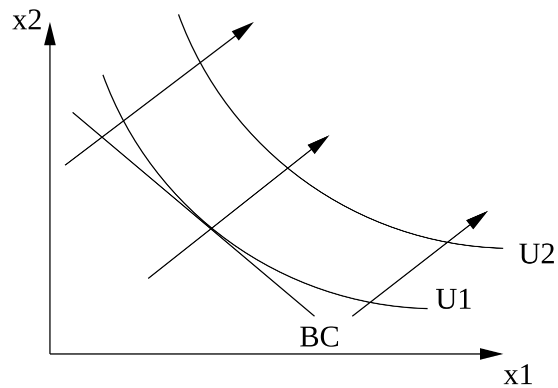
$$\frac{p_1}{p_2} = \frac{U_{x_1}}{U_{x_2}} \equiv MRS$$

where MRS stands for the marginal rate of substitution.

Note that we have two equations — the budget constraint as well as relative price equals MRS — in two unknows (x_1 and x_2). This system can be solved to obtain the so-called Marshallian or uncompensated (to understand why they are uncompensated wait until we see the compensated ones) **demand functions**:

$$x_1 = D_1(p_1, p_2, ..., I), \quad x_2 = D_2(p_1, p_2, ..., I), \quad \text{etc.}$$

graphically:



The slope of the indifference curves is the MRS (totally differentiating $\bar{U} = U(x_1, x_2)$ will tell you that) while the slope of the budget line is just the relative price p_1/p_2 (solve the budget constraint for x_2 to see this).

Some properties of the uncompensated demand functions:

- homogeneous of degree 0 in prices and income

- their income derivatives (or elasticities) satisfy:

$$\frac{p_1 x_1}{I}\varepsilon_{x_1,I} + \frac{p_2 x_2}{I}\varepsilon_{x_2,I} + ... = 1$$

Plug the uncompensated demand functions back into the utility function to obtain what is called the **indirect utility function**:

$$V(p_1, p_2, ..., I) \equiv U(D_1(p_1, p_2, ..., I), D_2(p_1, p_2, ..., I), ...)$$

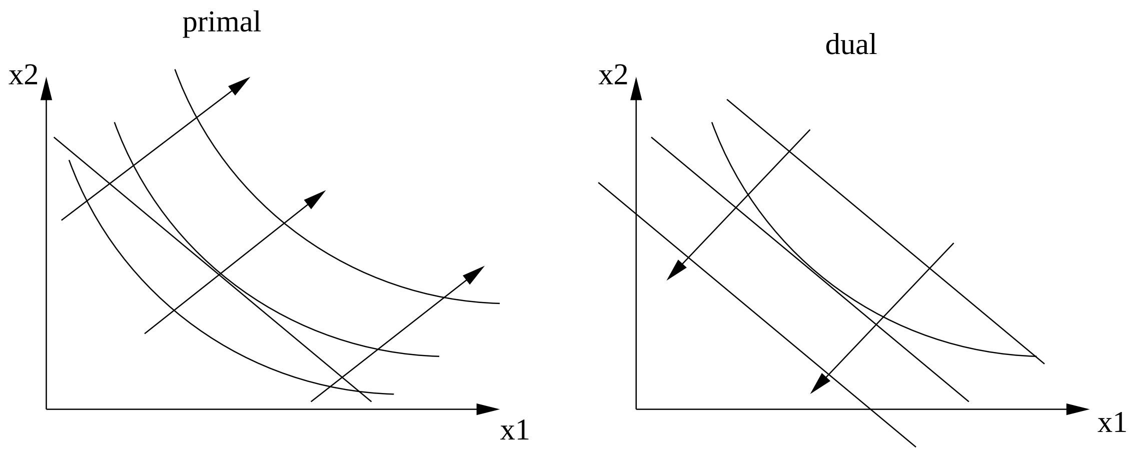
Some properties of the indirect utility function:

- homogeneous of degree 0 in prices and income.

- Roy's identity (for a proof see end of these notes):

$$D_i = -\frac{\delta V / \delta p_i}{\delta V / \delta I}$$

Expenditure Minimization



It is really getting at the *same* optimal point.

the dual problem:

$$\min_{x_1,x_2,...} \quad p_1x_1 + p_2x_2 + ...$$

subject to:

$$U(x_1, x_2, ...) = \bar{U}$$

We solve this constrained optimization problem by setting up the Lagrangean:

$$\mathcal{L}(x_1, x_2, ..., \lambda) = p_1 x_1 + p_2 x_2 + ... + \lambda(\bar{U} - U(x_1, x_2, ...))$$

FOCs:

$$\frac{\delta \mathcal{L}}{\delta x_1} = p_1 - \lambda U_{x_1} = 0$$

$$\frac{\delta \mathcal{L}}{\delta x_2} = p_2 - \lambda U_{x_2} = 0$$

dividing one by the other gives:

$$\frac{p_1}{p_2} = \frac{U_{x_1}}{U_{x_2}} \equiv MRS$$

which should look familiar — it's the same as for the primal.

Note that again we have two equations — the utility constraint as well as relative price equals MRS — in two unknows (x_1 and x_2). This system can be solved to obtain the so-called Hicksian or **compensated demand functions**:

$$x_1^c = D_1^c(p_1, p_2, ..., \bar{U}), \quad x_2^c = D_2^c(p_1, p_2, ..., \bar{U}), \quad \text{etc.}$$

Plugging these back into (the definition of) expenditure gives us the (minimal) **expenditure function**:

$$E(p_1, p_2, ..., \bar{U}) \equiv p_1 D_1^c(p_1, p_2, ..., \bar{U}) + p_2 D_2^c(p_1, p_2, ..., \bar{U}) + ...$$

This function tells us the expenditure needed to obtain a certain utility level at a given price vector.

Note that consumers' expenditure minimization closely resembles companies' cost minimization.

This is reflected in the following properties of the expenditure function:

- linear homogeneous in prices
- concave in prices
- Shephard's lemma:

$$\frac{\delta E(p_1, p_2, ..., \bar{U})}{\delta p_i} = D_i^c(p_1, p_2, ..., \bar{U})$$

Now, to demonstrate the usefulness of the dual approach, let us use it to prove Roy's identity — just as an example — in a few lines:

start with:

$$V(p_1, p_2, E(p_1, p_2, \bar{U})) = \bar{U}$$

differentiate with respect to (wrt) p_1 to get:

$$\frac{\delta V}{\delta p_1} + \frac{\delta V}{\delta I}\frac{\delta E}{\delta p_1} = 0$$

Note that $\delta E/\delta p_1 = x_1$ by Shephard's lemma and we have:

$$D_1(p_1, p_2, I) = -\frac{\delta V/\delta p_1}{\delta V/\delta I}$$

Q.E.D.

If you are still not convinced pls look up the primal proof of Slutsky's equation in the literature (Herberg does it in two pages I believe) and then look forward to the dual way b/c fortunately that is just a few lines.