Economics 51
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Solution Set: Homework 1

Question 1

a. We set the standard Lagrangian:
Change in notation: goods are x, y, z and prices are p, q, r

$$\max_{x,y,z} \mathcal{L} = x^\alpha y^\beta z^\gamma + \lambda(I - px - qy - rz)$$

FOC:

$$\frac{\partial L}{\partial x} = \alpha x^{\alpha-1} y^\beta z^\gamma - \lambda p \quad (1)$$
$$\frac{\partial L}{\partial y} = \beta x^\alpha y^{\beta-1} z^\gamma - \lambda q \quad (2)$$
$$\frac{\partial L}{\partial z} = \gamma x^\alpha y^\beta z^{\gamma-1} - \lambda r \qquad \text{etc.} \quad (3)$$

Although the algebra is annoying, it is not particularly difficult. The trick is to work in pairs: Equating 1 and 2 and then 1 and 3 gives:

$$x = y\left(\frac{q\alpha}{p\beta}\right) \qquad \text{and} \qquad x = z\left(\frac{r\alpha}{p\gamma}\right);$$

Re-writing for y and z and substituting into the budget function gives:

$$I = px\left(1 + \frac{\beta}{\alpha} + \frac{\gamma}{\alpha}\right)$$

So: $x = \frac{I}{p}\left(\frac{\alpha}{\alpha+\beta+\gamma}\right) \qquad \Rightarrow \qquad x = \frac{I\alpha}{p}$
Likewise $y = \frac{I\beta}{q}$ and $z = \frac{I\gamma}{r}$
Substituting directly into the Utility function gives the Indirect Utility Function :

$$V(p,q,r,I) = \left(\frac{I\alpha}{p}\right)^\alpha \left(\frac{I\beta}{q}\right)^\beta \left(\frac{I\gamma}{r}\right)^\gamma$$

Inverting, to solve for I, gives the Expenditure Function:

$$E(p,q,r,U) = U * \left(\frac{\alpha}{p}\right)^{-\alpha} \left(\frac{\beta}{q}\right)^{-\beta} \left(\frac{\gamma}{r}\right)^{-\gamma}$$

b. In this case, a graphical approach works better. $U(x_1, x_2) = min(x_1, x_2)$: Perfect complements
Looking at the diagram, we see that if for any budget line with a slope between the perfectly horizontal and vertical, tangency will occur at the corner of the utility curve.

Mathematically:

$$\frac{p_1}{p_2} \in (0, \infty) \Rightarrow x_1(p_1, p_2, m) = x_2(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

Plugging these demand functions into the utility function gives us the Indirect Utility Function:

$$V(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

The Expenditure Function is then:

$$E(p_1, p_2, U) = (p_1 + p_2) * U$$

If you think about it, it makes intuitive sense. We know that we will purchase the same amount of x_1 and x_2, we also know that $U = x_1 = x_2$. The rest follows directly.

c. Again, a graphical approach is simpler. $U(x_1, x_2) = x_1 + x_2$: Perfect Substitutes
When we contemplate the diagram we see that we will get corner solutions for our [uncompensated] demand functions, depending on the slope of the budget line. The intuition - since you are indifferent between the two goods (either gives the same level of utility,) you will buy whichever is cheaper.

Case 1. $\frac{p_1}{p_2} > 1$: x_2 cheaper
Here:

$$\begin{aligned} x_1(p_1, p_2, m) &= 0 \\ x_2(p_1, p_2, m) &= \frac{m}{p_2} \end{aligned}$$

Therefore:

$$\begin{aligned} V(p_1, p_2, m) &= \frac{m}{p_2} \\ E(p_1, p_2, U) &= p_2 * U \end{aligned}$$

Case 2. $\frac{p_1}{p_2} < 1$: x_1 cheaper
Here:

$$\begin{aligned} x_1(p_1, p_2, m) &= \frac{m}{p_1} \\ x_2(p_1, p_2, m) &= 0 \end{aligned}$$

Therefore:

$$\begin{aligned} V(p_1, p_2, m) &= \frac{m}{p_1} \\ E(p_1, p_2, U) &= p_1 * U \end{aligned}$$

Case 3. $\frac{p_1}{p_2} = 1$: equal cost

Here:
$$\begin{array}{rcl} x_1(p_1, p_2, m) & \in & \left[0, \frac{m}{p_1}\right] \\ x_2(p_1, p_2, m) & = & \frac{m - p_1 x}{p_2} \end{array}$$

In other words, I can buy any amount of x_1 that my income allows, any left over money will be spent on x_2. Of course we could reverse the formula, it would mean the same thing.

Moving on:
$$\begin{array}{rcccl} V(p_1, p_2, m) & = & \frac{m}{p_1} & = & \frac{m}{p_2} \\ E(p_1, p_2, U) & = & p_1 * U & = & p_2 * U \end{array}$$

Remember, $p_1 = p_2$ and x_1 and x_2 give the same utility. For the purposes of calculating the indirect utility function I can regard them as a single good costing a given price. All that is relevant is my income (m) and the cost of the “good”. The same thought process leads us to the expenditure function.

Question 2

$$U(x,y) = x - \frac{1}{y}$$

a. Setting up the Lagrangian:

$$\max_{x,y} \mathcal{L} = x - y^{-1} + \lambda(I - px - qy)$$

$$\text{FOC:} \quad \begin{aligned} \frac{\partial L}{\partial x} &= 1 - \lambda p & (4) \\ \frac{\partial L}{\partial y} &= y^{-2} - \lambda q & (5) \end{aligned} \quad \Rightarrow \quad \frac{1}{y^2} = \frac{q}{p} \qquad (6)$$

$$y = \sqrt{\left(\frac{p}{q}\right)} \quad \Rightarrow \quad x = \frac{I - \sqrt{pq}}{p}$$

b. 1. $y = p^{\frac{1}{2}} q^{-\frac{1}{2}} \quad \Rightarrow \quad \frac{\partial y}{\partial p} = \frac{1}{2} p^{-\frac{1}{2}} q^{-\frac{1}{2}} > 0$

2. $x = \frac{I}{p} - p^{-\frac{1}{2}} q^{\frac{1}{2}} \quad \Rightarrow \quad \frac{\partial x}{\partial q} = -\frac{1}{2} p^{-\frac{1}{2}} q^{-\frac{1}{2}} < 0$

c. We can represent this in the following diagram:

d. Although x is a gross complement for y, y is gross substitute for x. The presence of income effects accounts for this asymmetric result.

e. Now, include only substitution effect in our definition of complements and substitutes. This approach is equivalent to taking cross-derivatives of the compensated [Hicksian] demand functions. We know from the symmetry of the Slutsky Substitution Matrix that the cross-partials of Hicksian demand functions are equal.

$$V(p,q,m) = \frac{I}{p} - \sqrt{\frac{q}{p}} - \sqrt{\frac{q}{p}}$$

Inverting to get the expenditure function:

$$\begin{aligned} E(p,q,U) &= Vp + 2\sqrt{qp} && (7) \\ \frac{\partial E}{\partial q} &= \sqrt{\frac{p}{q}} && (8) \\ \frac{\partial^2 E}{\partial q \partial p} &= \frac{1}{2}\sqrt{\frac{1}{pq}} > 0 && (9) \end{aligned}$$

So, under this definition, x and y are net substitutes. Because of the symmetry of the Slutsky Matrix, we can't get the kind of asymmetic result that we saw in part (d.) with this approach. This result also holds for more than two goods.

Question 3

a. $U(x,y) = \alpha \ln(x-a) + \beta \ln(y-b)$

Where, α, a, β and b are all positive with $\alpha + \beta = 1$

We know from class that we can find the demand functions using

a. The Substitution method
b. The Langrangian method

Note: Some people maximized over the function $U(x,y) = (x-a)^\alpha (y-b)^\beta$. That's fine, when we answer questions concerning the expenditure function or compensated demand functions, replace $\exp U$ with plain U.

The Substitution Method

From the budget constraint: $y = \frac{m-px}{q}$
Unconstrained maximization over x:

$$\max_x U(x, \frac{m-px}{q}) = \alpha \ln(x-a) + \beta \ln(\frac{m-px}{q} - b)$$

FOC:

$$\frac{\alpha}{x-a} - \beta \frac{p}{q} \frac{1}{\frac{m-px}{q} - b} = 0$$

$$\alpha \frac{m-px}{q} - b\alpha = \beta \frac{p}{q} x - \frac{p}{q} \beta a$$

$$\alpha \frac{m}{q} + \frac{p}{q} \beta a - b\alpha = \beta \frac{p}{q} x + \alpha \frac{p}{q} x$$

$$x = \frac{q}{p} \frac{1}{\alpha + \beta} \left(\alpha \frac{m}{q} + \frac{p}{q} \beta a - b\alpha \right)$$

$$\Rightarrow x = \alpha \frac{m}{p} + \beta a - \frac{q}{p} b\alpha \qquad \text{since} \quad a + b = 1$$

$$\text{Given} \quad y = \frac{m-px}{q} = \frac{m}{q} - \frac{p}{q} x$$

$$\Rightarrow y = \beta \frac{m}{q} + b\alpha - \frac{p}{q} \beta a$$

Lagrangian Method

$$\max_{x,y,\lambda} L = U(x,y) + \lambda(m - px - qy)$$

FOC:

$$U_x - \lambda p = 0$$

$$U_y - \lambda q = 0$$

$$m - px - qy = 0 \qquad \text{budget constraint}$$

From the first two: $\frac{U_x}{U_y} = \frac{p}{q}$ where:

$$U_x = \alpha \frac{1}{x-a}$$

$$U_y = \beta \frac{1}{y-b}$$

$$\Rightarrow \qquad \text{MRS:} \quad \frac{U_x}{U_y} = \frac{\alpha}{\beta} \frac{y-b}{x-a} = \frac{p}{q}$$

From the budget constraint: $y = \frac{m-px}{q}$

So: $\frac{\alpha}{\beta} \frac{\frac{m-px}{q}}{-b} x - a = \frac{p}{q}$

After a lot of tiresome algebra we arrive at the demand functions:

$$x = \alpha \frac{m}{p} + \beta a - \frac{q}{p} b\alpha$$

and:

$$y = \beta \frac{m}{q} + b\alpha - \frac{p}{q} \beta a$$

Technically, you should check the second order conditions for the utility maximization (for the concavity of the utility functions) to make sure you have a maximum.

For those who went the extra step:

$$U_{xx} = -\alpha \frac{1}{(x-a)^2} < 0$$

$$U_{yy} = -\beta \frac{1}{(y-b)^2} < 0$$

Cross partials are zero.

b.

$$px = \alpha m + \beta a p - \alpha b q \qquad (10)$$
$$qy = \beta m + \alpha b q - \beta a p \qquad (11)$$

These are both linear functions of (p, q, m)

What interpretation can we attach to the parameters a and b? Notice that as the levels of x and y approach a and b respectively, utility becomes increasingly negative. We can therefore think of a and b as "subsistence levels" of x and y. In fact, we can rewrite the expenditure system as:

$$px = pa + \alpha(m - pa - qb) \quad (12)$$
$$qy = qb + \beta(m - pa - qb) \quad (13)$$

The first term stands for subsistence expenditure on goods x and y. The second term tells how much of the remaining income is spent additionally on x and y. x and y.

c. Preferences are homothetic if the marginal rates of substitution are constant along rays that extend through the origin. Another way of saying this is that the MRS is homogenous of degree zero, in (x, y).

$$MRS(\lambda x, \lambda y) = \frac{\alpha}{\beta}\frac{\lambda y - b}{\lambda x - a}$$

This is equal to $\lambda^0 MRS(x, y)$ for all $x > a$ and $y > b$ iff $a = b = 0$.

Alternative approaches include:

1. Showing that the Marshallian demands are linear in income. Note, it is not enough to show that the demand functions are homogeneous of order zero.
2. Showing that the utility function is homogeneous of order one in its components.

d. For the indirect utility function, we plug the demand functions back into the utility function.

$$V(p, q, m) = \alpha \ln (x(p, q, m)) + \beta (y(p, q, m)) \qquad \text{where} \quad a = b = 0$$

$$V(p, q, m) = \alpha \ln \alpha \frac{m}{p} + \beta \ln \beta \frac{m}{q} \quad (14)$$
$$= \alpha \ln \frac{\alpha}{p} + \beta \ln \frac{\beta}{q} + (\alpha + \beta) \ln m \qquad \text{where} \quad \alpha + \beta = 1 \quad (15)$$

To obtain the Expenditure Function, we invert the Indirect Utility Function to solve for m.

$$V = \alpha \ln \frac{\alpha}{p} + \beta \ln \frac{\beta}{q} + \ln m$$

$$\ln m = V - \alpha \ln \frac{\alpha}{p} - \beta \ln \frac{\beta}{q} \quad (16)$$
$$m = \exp(V - \alpha \ln \frac{\alpha}{p} - \beta \ln \frac{\beta}{q}) \quad (17)$$
$$m = \exp(U) \left(\frac{\alpha}{p}\right)^{-\alpha} \left(\frac{\beta}{q}\right)^{-\beta} \qquad \text{which is the same as ...} \quad (18)$$
$$m = \exp(U) \left(\frac{p}{\alpha}\right)^{\alpha} \left(\frac{q}{\beta}\right)^{\beta} \quad (19)$$

Sheppard's Lemma

$$x^c = E_p = \alpha \left(\frac{p}{\alpha}\right)^{\alpha-1} \left(\frac{q}{\beta}\right)^{\beta} \exp(U) * \frac{1}{\alpha} \quad (20)$$

$$= \exp(U) \left(\frac{p}{\alpha}\right)^{\alpha-1} \left(\frac{q}{\beta}\right)^{\beta} \quad (21)$$

$$= \exp(U) \left(\frac{\alpha q}{\beta p}\right)^{\beta} \qquad \text{from} \quad (\alpha + \beta = 1) \quad (22)$$

Likewise:

$$y^c = E_q = \exp(U) \left(\frac{\beta q}{\alpha q}\right)^{\alpha} \quad (23)$$

Expenditure Minimization

$$\min_{x,y} m = px + qy \quad s.t. \quad U = \alpha \ln(x) + \beta \ln(y)$$

Setting the following Langrangian:

$$\min_{x,y,\lambda} \mathcal{L} = px + qy + \lambda \left(U - \alpha \ln(x) - \beta \ln(y)\right)$$

$$\text{FOC: : } \begin{array}{ll} \frac{\partial L}{\partial x} & p - \lambda \frac{\alpha}{x} \\ \frac{\partial L}{\partial y} & q - \lambda \frac{\beta}{y} \end{array} \qquad \Rightarrow \qquad \begin{array}{rcl} \frac{px}{\alpha} & = & \frac{qy}{\beta} \quad (24) \\ x & = & \frac{\alpha q}{\beta p} y \quad (25) \end{array}$$

Using the third eq'n $\frac{\partial L}{\partial \lambda}$:

$$U = \alpha \ln \left(\frac{\alpha q}{\beta p} y\right) + \beta \ln y \quad (26)$$

$$= \ln \left(\frac{\alpha q}{\beta p}\right)^{\alpha} + (\alpha + \beta) \ln y \quad (27)$$

$$\exp(U) = \left(\frac{\alpha q}{\beta p}\right)^{\alpha} y \quad (28)$$

$$y = \exp(U) \left(\frac{\beta p}{\alpha q}\right)^{\alpha} \quad (29)$$

Then:

$$x = \left(\frac{\alpha q}{\beta p}\right) y \quad (30)$$

$$= \left(\frac{\beta p}{\alpha q}\right)^{\alpha-1} \exp U \quad (31)$$

$$= \left(\frac{\alpha q}{\beta p}\right)^{\beta} \exp U \quad (32)$$

Marshallian to Hicksian.

$$h(p,q,U) = x\left((p,q,E(p,q,U)\right) \quad (33)$$

$$x = \alpha \frac{m}{p} \quad (34)$$

$$x^c = \frac{\alpha}{p} * E(p,q,U) \quad (35)$$

$$= \exp(U)\left(\frac{p}{\alpha}\right)^{\alpha-1}\left(\frac{q}{\beta}\right)^{\beta} \quad (36)$$

$$= \exp(U)\left(\frac{\alpha}{p}\right)^{\beta}\left(\frac{q}{\beta}\right)^{\beta} \quad (37)$$

Likewise:

$$y = \beta\frac{m}{q} \quad (38)$$

$$y^c = \exp(U)\left(\frac{\beta p}{\alpha q}\right)^{\alpha} \quad (39)$$

Question 4

a. Looking at the diagram, we see that if you sold your house, you could make yourself better off even though B will be smaller than A. This answer doesn't depend on whether housing is a normal or inferior good. Why not? Although inferior goods have negative income effects ($\frac{\partial h}{\partial m} < 0$), the total price effect is still negative ($\frac{\partial h}{\partial p} < 0$.) With the price increase, we buy less housing. Moving to the left along the new budget line increases utility over point A.

b. Compare budget lines "B" and "C". They illustrate an income increase and so the direction of the income effect is now important when comparing houses B, C and A. If housing is inferior, the negative income effect implies $B < C < A$. If housing is a normal good then $C < B < A$. If housing were a Giffen good, with $\frac{\partial h}{\partial p} > 0$, then $B < A < C$ but we can see that the outcome is not possible under this budget constraint.

Question 5

a. In this example, the budget constraint is non-linear, with a spike at $l = T$.

b. Sharon prefers the childcare voucher if, given the voucher she would work a positive number of hours. Otherwise, the gov't would give her just enough to make her indifferent between working and staying home. Technically, if the $MRS_{l=T} < w(1-t)$ then Sharon would strictly prefer the voucher to the tax cut.

c. Notation: $U = c^{\frac{1}{2}} l^{\frac{1}{2}}$ where $c = (T - l)w\alpha$. Here, $\alpha = (1 - t)$

We're trying to solve the following problem:

Set the following unconstrained maximization:

$$\max_{l} U = ((T - l)w\alpha)^{1/2}\, l^{1/2}$$

This is equivalent to:

$$\max_{l} \ln U = \frac{1}{2} \ln T - l + \frac{1}{2} \ln w\alpha + \frac{1}{2} \ln l$$

So:

$$\frac{\partial \ln u}{\partial l} = -\frac{1}{2(T-l)} + \frac{1}{2l} \quad (40)$$

$$l = \frac{T}{2} \quad (41)$$

From the diagram, we see that working $\frac{T}{2}$ hours at $w\alpha$ gives the same utility as staying home and receiving R in non-wage income.

Equating utilities:

$$\left(\frac{T}{2} * w\alpha\right)^{1/2} \left(\frac{T}{2}\right)^{1/2} = \left(\frac{3}{8} * wT\right)^{1/2} \left(2 * \frac{T}{2}\right)^{1/2}$$

Equating coefficients: $\alpha = \frac{3}{2} \quad \Rightarrow \quad (1 - t) = 1.5$

Therefore, $t = +50\%$ or a subsidy of 50%.

Question 6

a. If the student, Chris, saves all their income:

$$5000(1 + .1) + 23100 = 28600$$

can be spent next year.

b. Chris can only borrow as much as can be repaid with next year's income. In other words, this year's income plus the Net Present Value (NPV) of next year's income can be spent now.

$$5000 + \frac{23100}{1 + 0.1} = 26000$$

c. This gives us the following budget constraint:

d. If the interest rate falls to 5%, the answer to parts a. and b. are now $5000(1 + .05) + 23100 = 28350$ and $5000 + \frac{23100}{1+0.05} = 27000$. This gives us the shallower budget constraint above. Note that the budget lines intersect at the endowment point,$E(5000, 23100)$. If you stay at that point (neither borrow not save,) the change in the interest rate has no effect on your utility.

e. Chris borrows \$5000 in year one at 10% interest . This is diagrammed as point $C_0 = (10000, 17600)$ [1]

f. A decrease in the interest rate to 5% results in the new consumption point of C_2. We separate the income and substitution effects by adding point C_1.

1. Substitution effect: $C_0 \to C_1$
2. Income effect: $C_1 \to C_2$

Remember that a decrease in the interest rate make borrowing easier and so the price of moving consumption forward falls. Its unlikely that the decrease in interest rate would lead Chris to borrow less, that would require present consumption to be really inferior.

[1] $17600 = 23100 - (5000 * 1.1)$