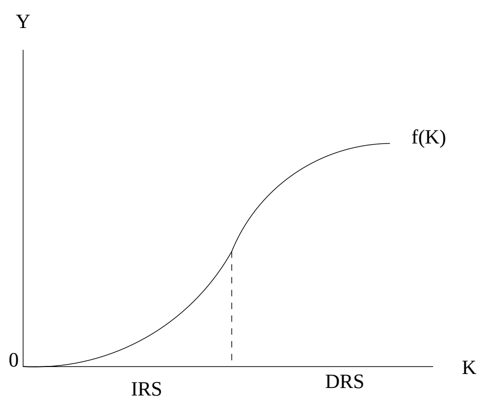
Economics 51
Summer 1999
TA: Joanna Campbell
jhcampbl@leland

Solution Set: Homework 2

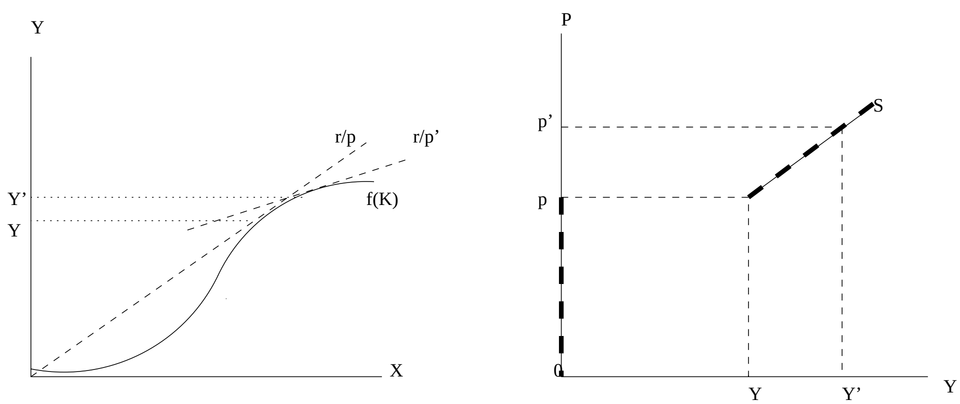
Question 1

a. See graph

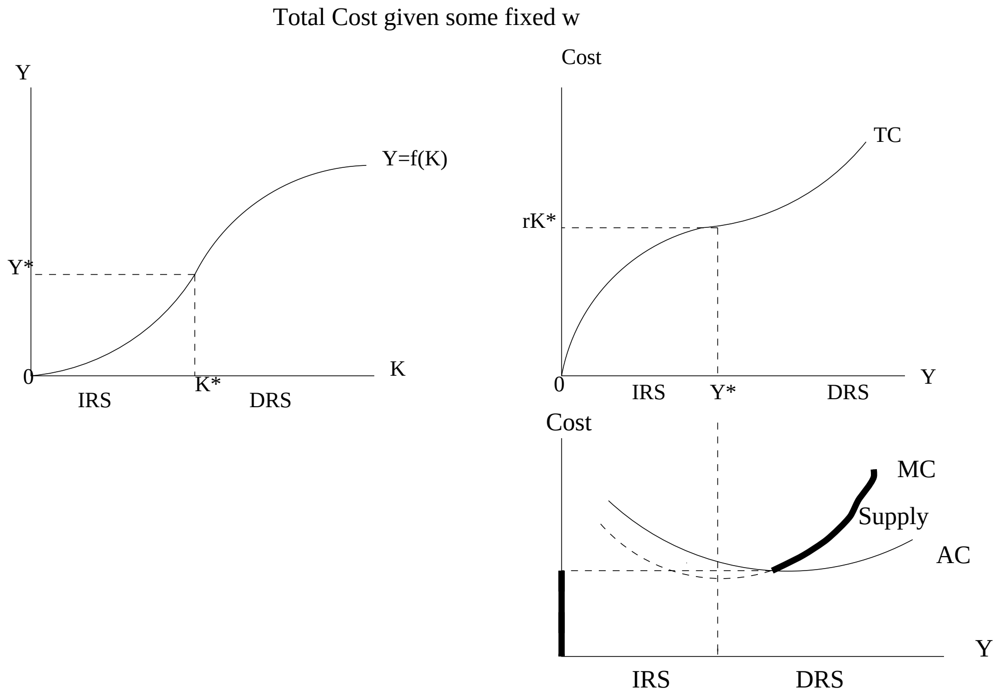


b.

Supply of Y, given some fixed w



c, d and e.



f. Yes, the two supply curves both represent the amount supplied at a given price by a profit maximizing, price taking firm.

Question 2

A firm operates under a Cobb-Douglas production function $Y = F(K, L) = K^\alpha L^\beta$; input factor prices r and w and an output price p.

a. Find conditions on α and β ensuring diminishing MP_K and MP_L.

$$MP_K : \frac{\partial F}{\partial K} = \alpha K^{\alpha-1} L^\beta$$
$$MP_L : \frac{\partial F}{\partial L} = \beta K^\alpha L^{\beta-1}$$

$$\frac{\partial MPK}{\partial K} = \alpha(\alpha-1)K^{\alpha-2}L^\beta < 0 \qquad \Leftrightarrow \qquad 0 < \alpha < 1$$

$$\frac{\partial MPK}{\partial L} = \beta(\beta-1)K^\alpha L^{\beta-2} < 0 \qquad \Leftrightarrow \qquad 0 < \beta < 1$$

Identifying the returns to scale:

$$Q(\lambda K, \lambda L) = (\lambda K)^\alpha (\lambda L)^\beta = \lambda^{\alpha+\beta} K^\alpha L^\beta$$

So, this production function is homogeneous of degree $\alpha + \beta$.

1. Increasing returns to scale iff $\alpha + \beta < 1$
2. Constant returns to scale iff $\alpha + \beta = 1$
3. Decreasing returns to scale iff $\alpha + \beta < 1$

b. Assuming $\alpha + \beta = 1$, the firm's problem is :

$$\min rK + wL \qquad s.t. \qquad \overline{Y} = K^{\alpha} L^{1-\alpha}$$

Setting the Lagrangian:

$$\mathcal{L} = rK + wL + \lambda(\overline{Y} - K^{\alpha} L^{1-\alpha})$$

$$\begin{aligned} \frac{\mathcal{L}}{\partial K} &: \ r - \lambda \alpha K^{\alpha-1} L^{1-\alpha} &= 0 \\ \frac{\mathcal{L}}{\partial L} &: \ w - \lambda(1-\alpha) K^{\alpha} L^{-\alpha} &= 0 \\ \frac{\mathcal{L}}{\partial \lambda} &: \ \overline{Y} - K^{\alpha} L^{1-\alpha} &= 0 \end{aligned}$$

From the first two FOCs:

$$\frac{r}{w} = \frac{\alpha}{1-\alpha} \cdot \frac{L}{K} \qquad \rightarrow \qquad \frac{L}{K} = \frac{1-\alpha}{\alpha} \cdot \frac{r}{w}$$

From the budget constraint:

$$\frac{\overline{Y}}{K} = \frac{L}{K}^{1-\alpha} \qquad \rightarrow \qquad \frac{\overline{Y}}{K} = \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{1-\alpha}$$

Therefore:

$$L = \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha} \cdot \overline{Y} \qquad \text{and} \qquad K = \overline{Y} \cdot \left(\frac{1-\alpha}{\alpha} \cdot \frac{r}{w}\right)^{\alpha-1}$$

$$C(r, w, Y) = r^{\alpha+1} \left(\frac{1-\alpha}{\alpha} \cdot \frac{1}{w}\right)^{\alpha} \cdot \overline{Y} + w^{-\alpha} \left(\frac{1-\alpha}{\alpha} \cdot r\right)^{\alpha-1} \cdot \overline{Y}$$

c. Now assume $\alpha + \beta < 1$. We can solve for the profit maximizing output in two ways.

<u>Alternative 1</u>: Solve the following for y,

$$\max_{y} \quad \Pi = pY - C(r, w, Y)$$

where, from above, we know

$$\begin{cases} C(r, w, Y) = r^{\alpha+1} \left(\frac{1-\alpha}{\alpha} \cdot \frac{1}{w}\right)^{\alpha} \cdot \overline{Y} + w^{-\alpha} \left(\frac{1-\alpha}{\alpha} \cdot r\right)^{\alpha-1} \cdot \overline{Y} \\ F(K, L) \ = K^{\alpha} L^{\beta} \end{cases}$$

Given decreasing returns to scale, we know that the solution will be

$$p - C'(Y) = 0 \quad \text{or} \quad P = MC, \qquad \Rightarrow \qquad Y(r, w, p)$$

This identifies our profit maximizing output as a function of p, r, w. We can then plug this into the conditional input demands from part b. to gain the unconditional input demands :

$$K\left(r, w, Y(r, w, p)\right) \quad \rightarrow \quad K(r, w, p)$$

.

<u>Alternative 2</u>: More directly, maximize the profit function with respect to K and L.

$$\max_{K,L} \quad \Pi = pY - rK - wL \quad s.t. \quad Y = F(K,L)$$

this will give you unconditional input demands to plug into the production function:

$$Y = F\left(K(r,w,p), L(r,w,p)\right) \quad \rightarrow \quad Y(r,w,p)$$

for the profit-maximizing supply.

e. Under increasing returns to scale, there is no profit-maximizing output, each additional unit contributes a larger profit margin.

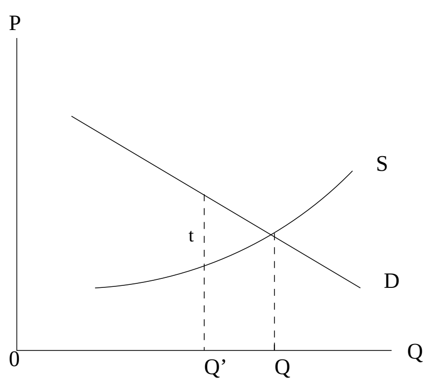
Question 3

a. If the lump-sum tax exceeds profits, it will drive the firm out of business.

b. A proportional tax simply reduces profits by a set percentage without influencing the optimal quantity produced.

c. A per-unit tax reduces the optimal quantity produced. The quantity supplied may drop to zero if net price received no longer covers the average cost.

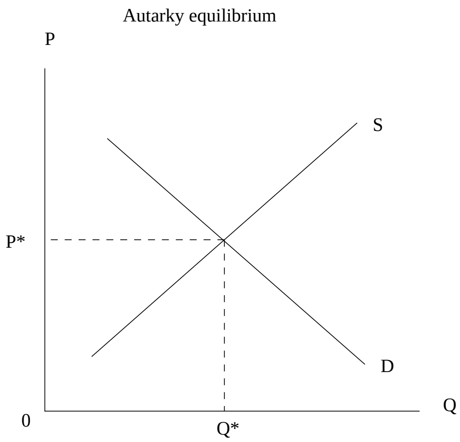
Per-unit tax on Output



Question 4

Aggregate Demand: $\alpha - \beta p$ Aggregate Supply: λp. In computing the following, we assume well-behaved individual demands.

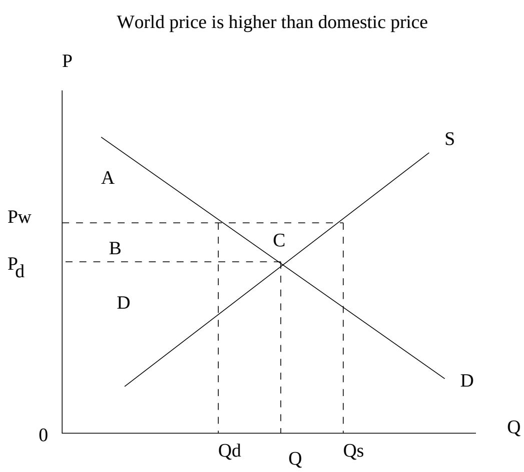
a. Under autarky (no international trade), the equilibrium price equates aggregate demand and supply, ensuring market clearing.



$$Q^S = Q^d \quad \rightarrow \quad \begin{aligned} \alpha - \beta p &= \gamma p \\ \alpha &= (\gamma+\beta)p \\ p &= \tfrac{\alpha}{\beta+\gamma} \end{aligned} \quad \Rightarrow \quad Q = \frac{\gamma\alpha}{\gamma+\beta}$$

b. Free trade at a fixed world price P_w

<u>Case 1</u>: $P_w > P$, domestic producers are relatively efficient and will export the good.



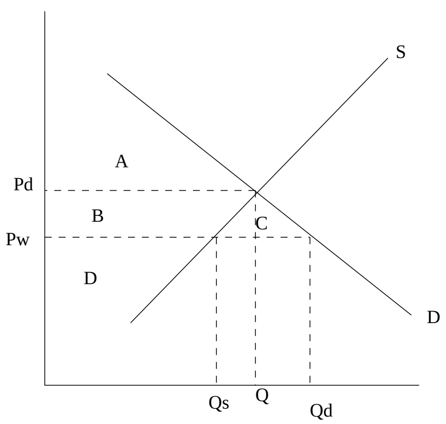
The new equilibrium: (P_W, Q_d) with exports of $E = Q_s - Q_d$

$$\begin{aligned} \triangle PC\text{:} &\quad +B+C \\ \triangle CS\text{:} &\quad -B \\ \text{NET CHANGE:} &\quad +C \end{aligned}$$

<u>Case 2</u>: $P_w = P$ no advantage to trade, no change in welfare compared with autarky case.

<u>Case 3</u>: $P_w < P$, domestic producers relatively inefficient - imports.

World Price lower than domestic price

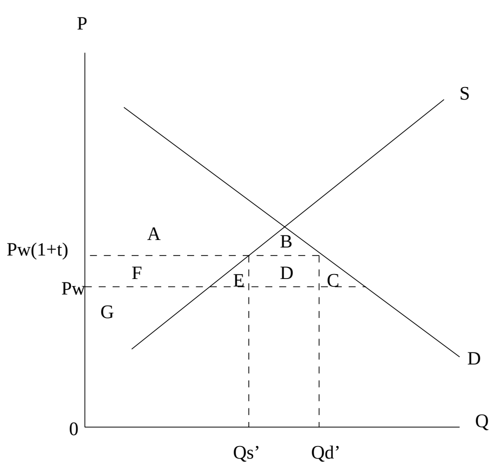


The new equilibrium:(P_W, Q_d) with imports of $E = Q_d - Q_s$

$$\begin{array}{lc} \triangle PC & -B \\ \triangle CS & +B + C \\ \text{NET CHANGE} & +C \end{array}$$

c. Imposing a tariff t on imports. Here, we calculate welfare changes using the free trade case as a baseline. The new equilibrium:$(P_W, Q_d^{'})$ with new import levels of $E^{'} = Q_d^{'} - Q_s^{'}$

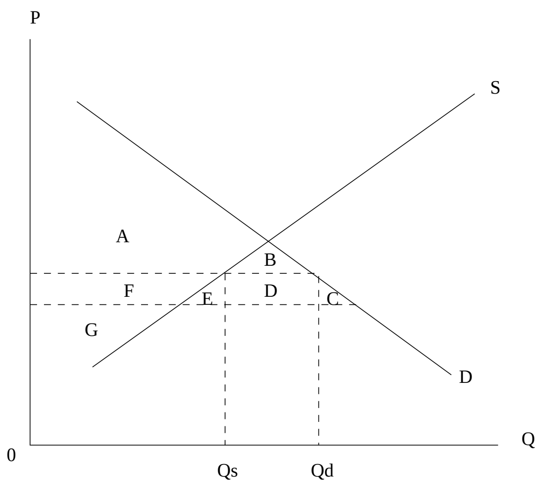
Tariff imposed on imports



$$\begin{array}{rl} \triangle PC\text{:} & +F \\ \triangle CS\text{:} & -F - E - D - C \\ \triangle\text{tariff revenue:} & +D \\ \text{DEAD WEIGHT LOSS:} & E \quad C \end{array}$$

d. Voluntary export restraints of amount E. The outcome for consumers and producers is the same as in part c. However, unless the government is auctioning import licenses as a source or revenue, the dead loss weight is larger. Therefore, from a social welfare point of view - a tariff is preferable to an equivalent VER.

Voluntary Export Restraints which approximate tariff

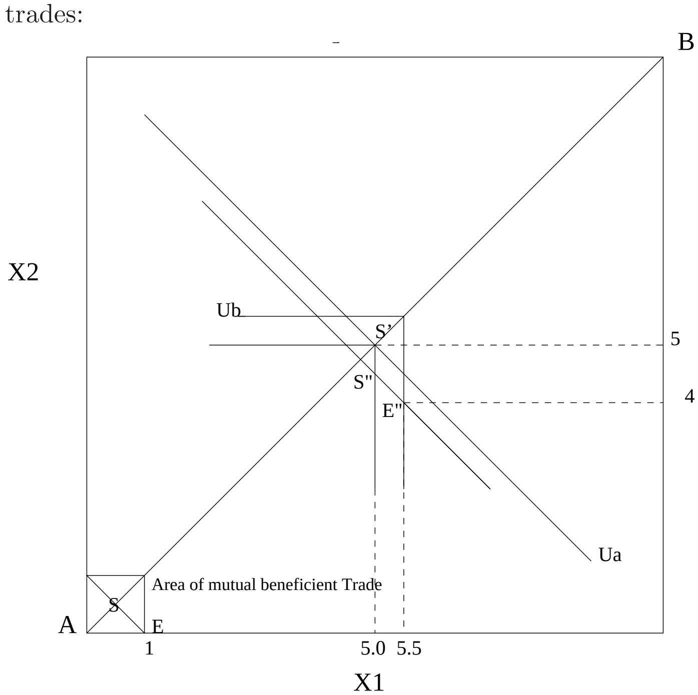


$$\begin{aligned} \triangle PC&: \quad +F \\ \triangle CS&: \quad -F-E-D-C \\ \text{DEAD WEIGHT LOSS}&: \quad E \quad D \quad C \end{aligned}$$

Question 5

Two goods, x_1 and x_2 are perfect substitutes for person A and perfect complements for person B. $E^A = (1, 0)$ and $E^B = (9, 10)$

a. The following Edgeworth box shows the endowments, indifference curves and area of mutually beneficial trades:



b. The market outcome is shown above as point S. The equilibrium price ratio is 1 (equalling the marginal rate of substitution of person A.) The equilibrium quantities are $S^A = (.5, .5)$ and $S^B = (9.5, 9.5)$ The area of mutually beneficial trades is contained in the triangle outlined by the indifference curves passing through endowment point E.

c. If you're concerned about an equitable outcome, you should pick allocation $i(5, 5)$. If you pick allocation ii the market outcome will settle on $S^A(4.75, 4.75)$ and $S^B(5.25, 5.25)$. Favoring person B however would lead you to pick the second allocation.

d. Allocation i from part c. exhibits Pareto efficiency, the allocation cannot be changed without making one of the agents worse off. It is therefore one of the points on the Contract curve, which is represented by the line $x_1 = x_2$. The original market outcome from part a. also lies on the Contract Curve.

e. Two potential endowments would be: $E^A(6, 4); E^B(4, 6)$ and $E^A(3, 7); E^B(7, 3)$. In fact any endowment lying along person A's indifference curve through the desired outcome would work. In each case, the equilibrium price ratio will be one.

Question 6

In this question, you are not supposed to do the social planner problem. Instead, solve the consumer and producer problems separately and then find the market clearing conditions.

$$\begin{aligned} U(C, L) &= \alpha \ln c + \beta \ln(1 - L) \\ Q &= L^\gamma \\ pC &= wL \end{aligned}$$

a. Solving the consumer problem:

$$\max_{C,L} \mathcal{L} = \alpha \ln c + \beta \ln 1 - L + \lambda(pC - \pi - wL)$$

from FOC:
$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C}: &\quad \frac{\alpha}{c} + p\lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial L}: &\quad \frac{-\beta}{1-L} - w\lambda = 0 \end{aligned}$$

$\Rightarrow$

$$\frac{1 - L}{C} \cdot \frac{\alpha}{\beta} = \frac{p}{w}$$

Using the budget constraint: $C = \frac{\pi}{p} - \frac{w}{p}$
After some algebra we end up with:

$$L = \left(\frac{\pi}{w} - \frac{\alpha}{\beta}\right)\left(1 - \frac{\alpha}{\beta}\right) \qquad \text{and} \qquad C = \frac{\pi}{p} - \frac{w}{p}\left(\frac{\pi}{w} - \frac{\alpha}{\beta}\right)\left(1 - \frac{\alpha}{\beta}\right)$$

as our utility maximizing coconut demand and labor supply.

b. We then do the firm problem:

$$\max_{L} paL^\gamma - wL$$

FOC: $a\gamma L^{\gamma-1} = \frac{w}{p}$

Again, the constraint in the maximization: $C = aL^{\gamma}$ identifies

$$L = \left(\frac{w}{ap\gamma}\right)^{\frac{1}{\gamma-1}} \qquad \text{and} \qquad C = a\left(\frac{w}{ap\gamma}\right)^{\frac{\gamma}{\gamma-1}}$$

c. To get the equilibrium output, we pick the price ratio that will equate the labor supply from both parts of the problem. Walras' Law ensures that the "coconut market" will also clear. This approach is basically equating the first order conditions (setting the $MRS = MP_L$.)

From part a.: $1 - L = \frac{p}{w} \cdot \frac{C\beta}{\alpha}$
From part b.: $C = aL^{\gamma}$ and $\frac{w}{p} = a\gamma L^{\gamma-1}$

Equating the price ratios $\frac{w}{p}$ from parts a. and b. (and substituting for C followed by some algebra) gives:

$$L = \frac{\alpha\gamma}{\beta + \alpha\gamma} \qquad \text{and then} \qquad C = a\left(\frac{\alpha\gamma}{\beta + \alpha\gamma}\right)^{\gamma}$$

The real wage is therefore:

$$\frac{w}{p} = a\gamma\left(\frac{\alpha\gamma}{\beta + \alpha\gamma}\right)^{\gamma-1}$$

An alternate approach

Use the equations for L and C from the producer approach to generate the optimal profit. Plug the profit function into the equations for L and C in the consumer problem.

We now have on the consumer side : $C_d(p, w)$ and $L_s(p, w)$
From the producer problem: $C_s(p, w)$ and $L_d(p, w)$
equating the two will give the equilibrium real wage which can then be plugged back into any of the above equations for the equilibrium Coconut and Labor amounts