Q1

a) $F(t, x, \dot{x}) = \sqrt{1+\dot{x}^2}$

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial \dot{x}} = \dot{x}(1+\dot{x}^2)^{-1/2}$$

Euler: $$0 - \frac{d}{dt}\left(\dot{x}(1+\dot{x}^2)^{-1/2}\right) = 0$$

integrate

$$\dot{x}(1+\dot{x}^2)^{-1/2} = C$$

$$\Rightarrow \dot{x} = D$$

$$x = Dt + E$$

use initial conditions

$$x(t_0) = Dt_0 + E = x_0$$

$$x(t_1) = Dt_1 + E = x_1$$

see next page

NB $$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = (1+\dot{x}^2)^{-3/2}\,\ddot{x} = 0$$

$$\Rightarrow \ddot{x} = 0$$

$$\dot{x} = C$$

then see above

$E = x_0 - Dt_0$

$Dt_1 + x_0 - Dt_0 = x_1$

$D = \frac{x_1 - x_0}{t_1 - t_0}$

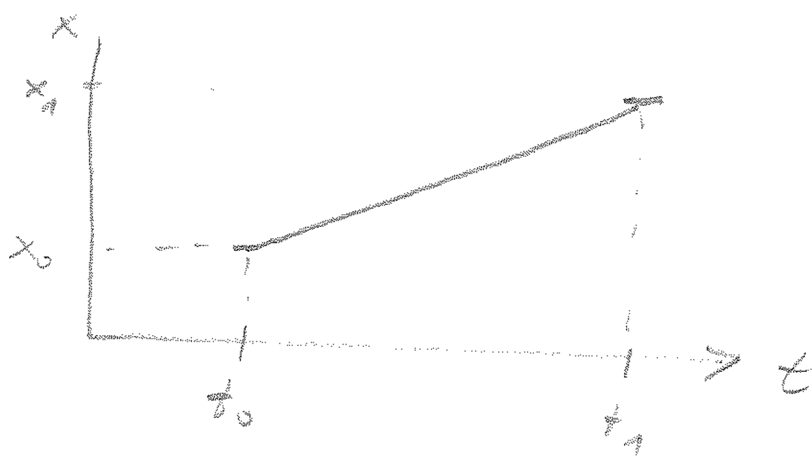
$$E = x_0 - t_0 \frac{x_1 - x_0}{t_1 - t_0}$$

$$= \frac{x_0 t_1 - x_0 t_0 - t_0 x_1 + t_0 x_0}{t_1 - t_0}$$

$$= \frac{t_1 x_0 - t_0 x_1}{t_1 - t_0}$$

$$\Rightarrow \quad x(t) = \frac{x_1 - x_0}{t_1 - t_0} t + \frac{t_1 x_0 - t_0 x_1}{t_1 - t_0}$$

b)



straight line

c)

the length of the graph from (t_0, x_0) to (t_1, x_1)

$\boxed{Q\ 2}$

$$F(t, x, \dot{x}) = e^{-t/10}\left(\frac{1}{100} t x - \dot{x}^2\right)$$

$$\frac{\partial F}{\partial x} = e^{-t/10} \frac{1}{100} t$$

$$\frac{\partial F}{\partial \dot{x}} = e^{-t/10} (-2) \dot{x}$$

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = -\frac{1}{10} e^{-t/10} (-2) \dot{x} + e^{-t/10} (-2) \ddot{x}$$

Euler: $\frac{1}{100} t - \frac{2}{10} \dot{x} + 2 \ddot{x} = 0$

$$\ddot{x} - \frac{1}{10} \dot{x} = -\frac{1}{200} t$$

$$u = \dot{x}$$

$$\dot{u} - \frac{1}{10} u = -\frac{1}{200} t$$

$$\dot{u} e^{-\frac{1}{10}t} - \frac{1}{10} u e^{-\frac{1}{10}t} = -\frac{1}{200} t e^{-\frac{1}{10}t}$$

$$u e^{-\frac{1}{10}t} = -\frac{1}{200} \int t e^{-\frac{1}{10}t} dt$$

$$\int t e^{-\frac{1}{10}t} dt = t(-10) e^{-\frac{1}{10}t} - \int (-10) e^{-\frac{1}{10}t} dt$$

$$= -10 t e^{-\frac{1}{10}t} - 100 e^{-\frac{1}{10}t} + \tilde{C}$$

$$u e^{-\frac{1}{10}t} = \frac{1}{20} t e^{-\frac{1}{10}t} + \frac{1}{2} e^{-\frac{1}{10}t} - \frac{1}{200} \tilde{C}$$

$$u = \frac{1}{20} t + \frac{1}{2} - \frac{1}{200} \tilde{C} e^{\frac{1}{10}t}$$

$$\dot{x} = \frac{1}{20} t + \frac{1}{2} - \frac{\tilde{C}}{200} e^{\frac{1}{10}t}$$

$$x = \frac{1}{40}t^2 + \frac{1}{2}t + C e^{\frac{1}{20}t} + D$$

initial conditions

$$x(0) = C + D = 0$$

$$x(T) = \frac{T^2}{40} + \frac{T}{2} + C e^{\frac{1}{20}T} - C = S$$

$$C\left(e^{\frac{1}{20}T} - 1\right) = S - \frac{T^2}{40} - \frac{T}{2}$$

$$C = \frac{S - T^2/40 - T/2}{e^{\frac{1}{20}T} - 1}$$

$$D = \frac{S - T^2/40 - T/2}{1 - e^{1/20\,T}}$$

$$\Rightarrow \quad x(t) = \frac{t^2}{40} + \frac{t}{2} + \frac{S - T^2/40 - T/2}{e^{1/20\,T} - 1} e^{\frac{1}{20}t}$$

$$+ \frac{S - T^2/40 - T/2}{1 - e^{1/20\,T}}$$

$\boxed{Q3}$

$$F(t, x, \dot{x}) = (10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}$$

$$\frac{\partial F}{\partial x} = (-2\dot{x} - 10x)e^{-t}$$

$$\frac{\partial F}{\partial \dot{x}} = (-2\dot{x} - 2x)e^{-t}$$

$$\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = (-2\ddot{x} - 2\dot{x})e^{-t} - (-2\dot{x} - 2x)e^{-t}$$

$$= (-2\ddot{x} + 2x)e^{-t}$$

Euler: $(-2\dot{x} - 10x)e^{-t} - (-2\ddot{x} + 2x)e^{-t} = 0$

$$2\ddot{x} - 2\dot{x} - 12x = 0$$

$$\ddot{x} - \dot{x} - 6 = 0$$

$$r^2 - r - 6 = 0$$

$$r_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = \frac{1}{2} \pm \frac{5}{2}$$

$$r_1 = -2 \qquad r_2 = 3$$

$$\Rightarrow \quad x = Ae^{-2t} + Be^{3t}$$

a) $x(0) = A + B = 0$

$$x(1) = Ae^{-2} + Be^{3} = 1$$

$$Ae^{-2} - Ae^{3} = 1$$

$$A = \frac{1}{e^{-2} - e^{3}} \qquad B = \frac{1}{e^{3} - e^{-2}}$$

$$\Rightarrow \quad x(t) = \frac{e^{-2t}}{e^{-2} - e^{3}} + \frac{e^{3t}}{e^{3} - e^{-2}}$$

b) $x(0) = 0 \quad \Rightarrow \quad A + B = 0$

transversality $\quad \left.\frac{\partial F}{\partial \dot{x}}\right|_T = 0$

$$\left.(-2\dot{x} - 2x)\, e^{-t}\right|_T = 0$$

$$-2\left(-2Ae^{-2T} + 3Be^{3T}\right)e^{-T}$$

$$-2\left(Ae^{-2T} + Be^{3T}\right)e^{-T} = 0$$

$$-Ae^{-2T} + 4Be^{3T} = 0$$

$$-Ae^{-2T} - 4Ae^{3T} = 0$$

$$A, B = 0$$