HW 10 solution

Q1

a)

$$\max \ \ln u_0 + \beta \ln u_1 + \beta^2 \ln u_2$$

s.t.

$$u_0 = A x_0^\alpha - x_1$$
$$u_1 = A x_1^\alpha - x_2$$
$$u_2 = A x_2^\alpha - x_3 = A x_2^\alpha$$

substitute

$$\max \ \ln (A x_0^\alpha - x_1) + \beta \ln (A x_1^\alpha - x_2) + \beta^2 \ln A x_2^\alpha$$

FOC wrt x_1 : $-\frac{1}{A x_0^\alpha - x_1} + \beta \frac{A \alpha x_1^{\alpha-1}}{A x_1^\alpha - x_2} = 0$ (1)

x_2 : $-\beta \frac{1}{A x_1^\alpha - x_2} + \beta^2 \frac{\alpha}{x_2} = 0$ (2)

solve : (2) $\Rightarrow$ $x_2 = \beta \alpha (A x_1^\alpha - x_2)$

$$x_2 = \frac{\beta \alpha A x_1^\alpha}{1 + \beta \alpha}$$

plug into (1)

$$A x_1^\alpha - x_2 = \beta A \alpha x_1^{\alpha-1} (A x_0^\alpha - x_1)$$

$$A x_1^\alpha - \frac{\beta \alpha A x_1^\alpha}{1 + \beta \alpha} = \beta \alpha A^2 x_1^{\alpha-1} x_0^\alpha - \beta \alpha A x_1^\alpha$$

$$1+\beta\alpha - \frac{\beta\alpha}{1+\beta\alpha} = \beta\alpha A x_0^{\alpha} / x_1$$

$$x_1 = \frac{\beta\alpha A x_0^{\alpha}}{1+\beta\alpha - \frac{\beta\alpha}{1+\beta\alpha}} = \frac{\beta\alpha(1+\beta\alpha)Ax_0^{\alpha}}{(1+\alpha\beta)(1+\beta\alpha)-\beta\alpha}$$

$$= \frac{\beta\alpha(1+\beta\alpha)Ax_0^{\alpha}}{1+\beta\alpha+\beta^2\alpha^2} = \frac{\beta\alpha(1+\beta\alpha)Ax_0^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}$$

$$x_2 = \frac{\beta\alpha A x_1^{\alpha}}{1+\beta\alpha} = \frac{\beta\alpha A\left(\frac{\beta\alpha(1+\beta\alpha)Ax_0^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}\right)^{\alpha}}{1+\alpha\beta}$$

$$= \frac{(\beta\alpha)^{1+\alpha}A^{1+\alpha}(1+\beta\alpha)^{\alpha-1}x_0^{\alpha^2}}{(1+\beta\alpha(1+\alpha\beta)^{\alpha}}$$

$$u_0 = Ax_0^{\alpha} - x_1 = \frac{Ax_0^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}$$

$$u_1 = Ax_1^{\alpha} - x_2 = A\left(\frac{\beta\alpha(1+\beta\alpha)Ax_0^{\alpha}}{1+\beta\alpha(1+\alpha\beta)}\right)^{\alpha} - x_2$$

$$= \frac{A^{\alpha+1}(\beta\alpha)^{\alpha}(1+\beta\alpha)^{\alpha}x_0^{\alpha^2}}{(1+\beta\alpha(1+\alpha\beta))^{\alpha}} - x_2$$

$$= \frac{(1-\alpha\beta)A^{\alpha+1}(\beta\alpha)^{\alpha}(1+\beta\alpha)^{\alpha}x_0^{\alpha^2}}{(1+\beta\alpha(1+\alpha\beta))^{\alpha}}$$

$$u_2 = Ax_2^{\alpha} = \frac{A(\beta\alpha)^{(1+\alpha)\alpha}A^{(1+\alpha)\alpha}(1+\beta\alpha)^{(\alpha-1)\alpha}x_0^{\alpha^3}}{(1+\beta\alpha(1+\beta\alpha))^{\alpha^2}}$$

b) $t=2$

$$\max_{u_2} \beta^2 \ln u_2$$
$$\text{s.t. } u_2 = A x_2^\alpha$$

$$\Rightarrow \underline{\underline{u_2}} = A x_2^\alpha$$

$$\underline{\underline{J_2}} = \beta^2 \ln A x_2^\alpha$$

$t=1$ $$\max_{u_1} \beta \ln u_1 + \beta^2 \ln A(Ax_1^\alpha - u_1)^\alpha$$

FOC $$\frac{\beta}{u_1} - \beta^2 \frac{\alpha}{Ax_1^\alpha - u_1} = 0$$

$$Ax_1^\alpha - u_1 = \beta\alpha u_1$$

$$\underline{\underline{u_1}} = \frac{Ax_1^\alpha}{1+\beta\alpha}$$

$$\underline{\underline{J_1}} = \beta \ln \frac{Ax_1^\alpha}{1+\beta\alpha} + \beta^2 \underbrace{\ln A\left(Ax_1^\alpha - \frac{Ax_1^\alpha}{1+\beta\alpha}\right)^\alpha}_{\underbrace{\ln A + \alpha \ln \frac{\beta\alpha}{1+\beta\alpha} A x_1^\alpha}_{\ln A + \alpha \ln \frac{\beta\alpha}{1+\beta\alpha} A + \alpha^2 \ln x_1}}$$

$t=0$

$$\max_{u_0} \ln u_0 + \beta\left(\ln \frac{A}{1+\beta\alpha} + \alpha \ln x_1\right)$$
$$+ \beta^2\left(\ln A + \alpha \ln \frac{\beta\alpha}{1+\beta\alpha} A + \alpha^2 \ln x_1\right)$$

with $x_1 = Ax_0^\alpha - u_0$

FOC $\quad \frac{1}{u_0} - \frac{\beta\alpha}{Ax_0^\alpha - u_0} - \frac{\beta^2\alpha^2}{Ax_0^\alpha - u_0} = 0$

$$Ax_0^\alpha - u_0 = \beta\alpha(1+\beta\alpha)u_0$$

$$\underline{\underline{u_0}} = \frac{Ax_0^\alpha}{1+\beta\alpha(1+\beta\alpha)}$$

$$\underline{\underline{V_0}} = \ln \frac{Ax_0^\alpha}{1+\beta\alpha(1+\beta\alpha)} + \beta\left(\ln\frac{A}{1+\beta\alpha} + \alpha\ln\left(Ax_0^\alpha - \frac{Ax_0^\alpha}{1+\beta\alpha(1+\beta\alpha)}\right)\right) +$$

$$\beta^2\left(\ln A + \alpha\ln\frac{\beta\alpha}{1+\beta\alpha}A + \alpha^2\ln\left(\underbrace{Ax_0^\alpha - \frac{Ax_0^\alpha}{1+\beta\alpha(1+\alpha\beta)}}_{\frac{\beta\alpha Ax_0^\alpha}{1+\beta\alpha(1+\alpha\beta)}}\right)\right)$$

(try to find general rule

$$u_{T-k} = \frac{Ax_{T-k}^\alpha}{\sum_{s=0}^{k}(\alpha\beta)^s}$$

(ie $1+\alpha\beta+\alpha\beta^2+\ldots$

note: as $k \to \infty$

$$u = \frac{Ax^\alpha}{1} = (1-\alpha\beta)Ax^\alpha$$)

c)

$$x_1 = A x_0^\alpha - u_0 = A x_0^\alpha - \frac{A x_0^\alpha}{1+\beta\alpha(1+\alpha\beta)}$$

$$= \frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} A x_0^\alpha$$

same as in a)

$$x_2 = A x_1^\alpha - u_1 = A x_1^\alpha - \frac{A x_1^\alpha}{1+\alpha\beta}$$

$$= \frac{\alpha\beta \; A x_1^\alpha}{1+\alpha\beta}$$

$$= \frac{\alpha\beta \; A \left(\frac{\beta\alpha(1+\alpha\beta)}{1+\beta\alpha(1+\alpha\beta)} A x_0^\alpha \right)^\alpha}{1+\alpha\beta}$$

$$= \frac{(\alpha\beta)^{1+\alpha} A^{1+\alpha} (1+\alpha\beta)^{\alpha-1} x_0^{\alpha^2}}{(1+\beta\alpha(1+\alpha\beta))^\alpha}$$

same as in a)

Q1

max $(3-u_t)x_t^2$ $\quad u_t \in [0,1]$

$s=T$: $u_T = 0$

$J_T = 3x_T^2$

$s=T-1$: max $(3-u)x^2 + 3ux$

FOC $-x^2 + 3x$

$\boxed{Q2}$

a) max $-e^{-u} - \frac{1}{2}e^{-x} + \beta(-\alpha)e^{-2x+u}$

FOC $e^{-u} - \alpha\beta e^{-2x+u} = 0$

$$e^{-u} = \alpha\beta e^{-2x} e^{u}$$

$$1 = \alpha\beta e^{-2x} e^{2u}$$

$$\frac{e^{2x}}{\alpha\beta} = e^{2u}$$

$$2x - \ln\alpha\beta = 2u$$

$$u = x - \frac{1}{2}\ln\alpha\beta$$

plug in

$$-e^{-x+\frac{1}{2}\ln\alpha\beta} - \frac{1}{2}e^{-x} - \alpha\beta e^{-2x+x-\frac{1}{2}\ln\alpha}$$

$$-e^{-x}\sqrt{\alpha\beta} - \frac{1}{2}e^{-x} - \alpha\beta e^{-x}\frac{1}{\sqrt{\alpha\beta}}$$

$$= -\underbrace{\left(2\sqrt{\alpha\beta} + \frac{1}{2}\right)}_{\alpha} e^{-x}$$

$$\alpha = 2\sqrt{\alpha\beta} + \frac{1}{2}$$

$$\alpha - 2\sqrt{\beta}\sqrt{\alpha} - \frac{1}{2} = 0$$

$$(\sqrt{\alpha} - \sqrt{\beta})(\sqrt{\alpha} - \sqrt{\beta})$$

$$\alpha - 2\sqrt{\beta}\sqrt{\alpha} + \beta - \beta - \frac{1}{2} = 0$$

$$(\sqrt{\alpha} - \sqrt{\beta})^2 = \beta + \frac{1}{2}$$

$$\sqrt{\alpha} = \sqrt{\beta + \frac{1}{2}} + \sqrt{\beta}$$

$$\alpha = \beta + \frac{1}{2} + 2\sqrt{\beta(\beta + \frac{1}{2})} + \beta$$

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c) $$x_{t+1} = 2x_t - u_t = 2x_t - x_t + \frac{1}{2}\ln\alpha\beta$$

$$= x_t + \frac{1}{2}\ln\alpha\beta$$

$$x_t = x_0 + \frac{t}{2}\ln\alpha\beta$$