Econ 300
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Answers to Problem Set 4

Problem 1

The easiest way to find out if a production function has increasing, decreasing, or constant returns to scale is to multiply each input in the function with a positive constant, ($t > 0$), and then see if the whole production function is multiplied with a number that is higher, lower, or equal to that constant.

You can determine if the marginal product of an input is increasing, decreasing, or constant by looking how the MP reacts to a change in that input. That is easiest to find out by taking a derivative of the marginal product with respect to the input in question.

The definition of technical rate of substitution is $MRTS_{L,K} = \frac{MP_L}{MP_K}$.

(a) $f(L,K) = L + 2K$ The production function has constant returns to scale.

$$\begin{aligned} f(tL, tK) &= tL + 2tK \\ &= t(L + 2K) \\ &= tf(L, K). \end{aligned}$$

The marginal product of labor is constant, since

$$MP_L = \frac{\partial f(L,K)}{\partial L} = 1.$$

(If you take a derivative of this with respect to L, you get 0). The same applies to marginal product of capital:

$$MP_K = \frac{\partial f(L,K)}{\partial K} = 2$$

is constant in K.

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{1}{2}.$$

(b) $f(L,K) = \frac{1}{5}LK^2$ The production function has increasing returns to scale.

$$\begin{aligned} f(tL, tK) &= \frac{1}{5}(tL)(tK)^2 \\ &= \frac{1}{5}t^3 LK^2 \\ &= t^3 f(L, K). \end{aligned}$$

The marginal product of labor is constant

$$\begin{aligned} MP_L &= \frac{\partial f(L,K)}{\partial L} \\ &= \frac{K^2}{5}. \end{aligned}$$

The marginal product of capital is increasing

$$\begin{aligned} MP_K &= \frac{\partial f(L,K)}{\partial K} \\ &= \frac{2L}{5}K \quad \text{and} \\ \frac{\partial MP_K}{\partial K} &= \frac{2L}{5} > 0. \end{aligned}$$

$$MRTS_{L,K} = \frac{MP_L}{MP_K} = \frac{K}{2L}.$$

(c) $f(L,K) = L^{\frac{1}{4}}K^{\frac{3}{4}}$ This function has constant returns to scale.

$$\begin{aligned} f(tL,tK) &= (tL)^{\frac{1}{4}}(tK)^{\frac{3}{4}} \\ &= tL^{\frac{1}{4}}K^{\frac{3}{4}} \\ &= tf(L,K). \end{aligned}$$

Marginal product of labor is diminishing.

$$\begin{aligned} MP_L &= \frac{1}{4}L^{-\frac{3}{4}}K^{\frac{3}{4}} \quad \text{and} \\ \frac{\partial MP_L}{\partial L} &= -\frac{3}{16}L^{-\frac{7}{4}}K^{\frac{3}{4}} < 0. \end{aligned}$$

Marginal product of capital is decreasing in K.

$$\begin{aligned} MP_K &= \frac{3}{4}L^{\frac{1}{4}}K^{-\frac{1}{4}} \quad \text{and} \\ \frac{\partial MP_K}{\partial K} &= -\frac{3}{16}L^{\frac{1}{4}}K^{-\frac{5}{4}} < 0. \end{aligned}$$

$$MRTS_{L,K} = \frac{\frac{1}{4}L^{-\frac{3}{4}}K^{\frac{3}{4}}}{\frac{3}{4}L^{\frac{1}{4}}K^{-\frac{1}{4}}} = \frac{K}{3L}.$$

(d) $f(L,K) = \left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^3$ The production function has constant returns to scale.

$$\begin{aligned} f(tL,tK) &= \left((tL)^{\frac{1}{3}} + (tK)^{\frac{1}{3}}\right)^3 \\ &= \left[t^{\frac{1}{3}}\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)\right]^3 \\ &= t\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^3 \\ &= tf(L,K). \end{aligned}$$

MP_L decreases in L.

$$\begin{aligned} MP_L &= 3\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^2 \frac{1}{3} L^{-\frac{2}{3}} \\ &= \left(L^{\frac{2}{3}} + 2L^{\frac{1}{3}}K^{\frac{1}{3}} + K^{\frac{2}{3}}\right) L^{-\frac{2}{3}} \\ &= 1 + 2L^{-\frac{1}{3}}K^{\frac{1}{3}} + L^{-\frac{2}{3}}K^{\frac{2}{3}} \quad \text{and} \\ \frac{\partial MP_L}{\partial L} &= -\frac{2}{3}L^{-\frac{4}{3}}K^{\frac{1}{3}} - \frac{2}{3}L^{-\frac{5}{3}}K^{\frac{2}{3}} < 0. \end{aligned}$$

MP_K decreases in K.

$$\begin{aligned} MP_K &= 3\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^2 \frac{1}{3} K^{-\frac{2}{3}} \\ &= \left(L^{\frac{2}{3}} + 2L^{\frac{1}{3}}K^{\frac{1}{3}} + K^{\frac{2}{3}}\right) K^{-\frac{2}{3}} \\ &= 1 + 2L^{\frac{1}{3}}K^{-\frac{1}{3}} + L^{\frac{2}{3}}K^{-\frac{2}{3}} \quad \text{and} \\ \frac{\partial MP_K}{\partial K} &= -\frac{2}{3}L^{\frac{2}{3}}K^{-\frac{5}{3}} - \frac{2}{3}L^{\frac{1}{3}}K^{-\frac{4}{3}} < 0. \end{aligned}$$

$$\begin{aligned} MRTS_{L,K} &= \frac{3\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^2 \frac{1}{3}L^{-\frac{2}{3}}}{3\left(L^{\frac{1}{3}} + K^{\frac{1}{3}}\right)^2 \frac{1}{3}K^{-\frac{2}{3}}} \\ &= \left(\frac{K}{L}\right)^{\frac{2}{3}} \end{aligned}$$

Problem 2

(a) The minimization problem

$$\min wL + rK \quad \text{s.t.} \quad Q = AL^{\alpha}K^{\beta}$$

can be written as a Lagrangean

$$\Lambda = wL + rK - \lambda(AL^{\alpha}K^{\beta} - Q).$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \Lambda}{\partial L} &= w - \lambda\alpha AL^{\alpha-1}K^{\beta} = 0 \\ \frac{\partial \Lambda}{\partial K} &= w - \lambda\beta AL^{\alpha}K^{\beta-1} = 0 \\ \frac{\partial \Lambda}{\partial \lambda} &= -AL^{\alpha}K^{\beta} + Q = 0. \end{aligned}$$

We can modify the first equation to have

$$\lambda = \frac{w}{\alpha AL^{\alpha-1}K^{\beta}}$$

Using this in the second equation gives

$$
\begin{aligned}
r &= \frac{w\beta A L^{\alpha} K^{\beta-1}}{\alpha A L^{\alpha-1} K^{\beta}} = \frac{w\beta L}{\alpha K} \\
\iff \frac{wL}{\alpha} &= \frac{rK}{\beta}.
\end{aligned}
$$

Since the $MRTS_{L,K}$ for this function only depends on the ratio of K to L, the expansion path is linear. (Or more generally, the expansion path is linear for homothetic functions).

(b) With $A = 2, \alpha = \beta = \frac{1}{2}$, and $K = \bar{K}$, we have

$$Q = 2L^{\frac{1}{2}}\bar{K}^{\frac{1}{2}}$$

Now

$$\min wL + r\bar{K} \quad \text{s.t.} \quad Q = 2L^{\frac{1}{2}}\bar{K}^{\frac{1}{2}}$$

Implies

$$L^{\frac{1}{2}} = \frac{Q}{2\bar{K}^{\frac{1}{2}}} \implies L = \frac{Q^2}{4\bar{K}}$$

The short run cost function is

$$c(w, r, Q, \bar{K}) = w\frac{Q^2}{4\bar{K}} + r\bar{K}.$$

(c) To find the optimal amount of capital, we look at the firm's cost minimization problem

$$\min wL + rK \quad \text{s.t.} \quad Q = 2L^{\frac{1}{2}}K^{\frac{1}{2}},$$

and build a Lagrangean

$$\Lambda = wL + rK - \lambda(2L^{\frac{1}{2}}K^{\frac{1}{2}} - Q).$$

From the first order conditions

$$
\begin{aligned}
\frac{\partial \Lambda}{\partial L} &= w - \lambda L^{-\frac{1}{2}}K^{\frac{1}{2}} = 0, \\
\frac{\partial \Lambda}{\partial K} &= r - \lambda L^{\frac{1}{2}}K^{-\frac{1}{2}} = 0, \quad \text{and} \\
\frac{\partial \Lambda}{\partial \lambda} &= -2L^{\frac{1}{2}}K^{\frac{1}{2}} + Q = 0
\end{aligned}
$$

we can solve λ (using the first equation) as

$$\lambda = \frac{w}{L^{-\frac{1}{2}}K^{\frac{1}{2}}}.$$

Using this in the second equation gives us

$$
\begin{aligned}
r = \frac{wL^{\frac{1}{2}}K^{-\frac{1}{2}}}{L^{-\frac{1}{2}}K^{\frac{1}{2}}} &= \frac{wL}{K} \\
\implies L = \frac{r}{w}K.
\end{aligned}
$$

Using this in the third equation gives us the optimal K:

$$
\begin{aligned}
Q = 2\left(\frac{r}{w}K\right)^{\frac{1}{2}} K^{\frac{1}{2}} &= 2\left(\frac{r}{w}\right)^{\frac{1}{2}} K \\
\implies K &= \frac{1}{2}\left(\frac{w}{r}\right)^{\frac{1}{2}} Q.
\end{aligned}
$$

We can also see that the optimal amount of labor is

$$L = \frac{r}{w}K = \frac{1}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}} Q.$$

(d) The long run total cost function tells us the minimal costs needed to produce Q units of output:

$$\begin{aligned} c(w, r, Q) &= wL(w, r, Q) + rK(w, r, Q) \\ &= w\frac{1}{2}\left(\frac{r}{w}\right)^{\frac{1}{2}} Q + r\frac{1}{2}\left(\frac{w}{r}\right)^{\frac{1}{2}} Q \\ &= (wr)^{\frac{1}{2}} Q. \end{aligned}$$

Question 3

a) Problem: Max (L, K): p $L^{1/3} K^{1/3}$ – wL – rK

FOC:

$\partial\pi / \partial L = p\ 1/3\ L^{-2/3} K^{1/3} = w$
$\partial\pi / \partial K = p\ 1/3\ L^{1/3} K^{-2/3} = r$

The first equation gives: $L = \dfrac{p^3}{27w^2 r}$

Plug into the second equation: $\dfrac{p}{3} L^{1/3} \dfrac{p^2}{9w^2 L^{4/3}} = r$

Then, $L = \dfrac{p^3}{27w^2 r}$

And $K^{1/3} = \dfrac{3w(\frac{p^3}{27w^2 r})^{2/3}}{p}$ so $K = \dfrac{p^3}{27r^2 w}$

Finally, $Q = L^{1/3}(\frac{w}{r}L)^{1/3}$ and $Q_a{*} = \dfrac{p}{2w} + rQ^{3/2}(\frac{w}{r})^{1/2} = 2Q^{3/2}(wr)^{1/2}$

b) The problem is Max (Q, L, K): pQ – wL – rK s.t. Q = $L^{1/3} K^{1/3}$

Then, £ = pQ – wL – rK + λ (Q - $L^{1/3} K^{1/3}$)

FOC:

$\partial£ / \partial L = w - \lambda\ 1/3\ L^{-2/3} K^{1/3} = 0$
$\partial£ / \partial K = r - \lambda\ 1/3\ L^{1/3} K^{-2/3} = 0$
$\partial£ / \partial \lambda = Q - L^{1/3} K^{1/3} = 0$

From 1 and 2: w/r = K/L

Then $Q = L^{1/3}(\frac{w}{r}L)^{1/3}$ and $L{*} = Q^{3/2}(\frac{r}{w})^{1/2}$ and $K{*} = Q^{3/2}(\frac{w}{r})^{1/2}$

And the Cost function is: $C{*}(Q) = w(\frac{Q^2}{25} + 4\frac{Q^2}{25}) = \frac{wQ^2}{25} + rQ^{3/2}(\frac{w}{r})^{1/2} = 2Q^{3/2}(wr)^{1/2}$

c) The problem is Max (Q): pQ – C(Q)

FOC: $\partial\pi / \partial Q = p - MC = 0$

From C(Q): $MC = 3\,(wr)^{1/2}\,Q^{1/2}$

Then $p = 3\,(wr)^{1/2}\,Q^{1/2}$ and $Q^* = (\frac{p^2}{9wr})$

Plug this Q* into the conditional factor demands of point b to get:

$L^* = \frac{p^3}{27w^2r}$ and $K^* = \frac{p^3}{27r^2w}$

<u>Question 4</u>

a) *Cost minimization*

For firm A: $Q_a = L_a^{1/2}$ so $L_a = Q_a^2$ and $C(L_a) = w\,Q_a^2$

For firm B: $Q_b = 2\,L_b^{1/2}$ so $L_b = Q_b^2 / 4$ and $C(L_b) = w\,Q_b^2 / 4$

Profit maximization:

<u>For firm A</u>: Max (Q_a): $p\,Q_a – w\,Q_a^2$

FOC:

$\partial\pi / \partial Q_a = p – 2\,w\,Q_a = 0$ so $Q_a^* = \frac{p}{2w}$

<u>For firm B</u>: Max (Q_b): $p\,Q_b – w\,Q_b^2 / 4$

FOC:

$\partial\pi / \partial Q_b = p – 2\,w\,Q_a = 0$ so $Q_b^* = \frac{2p}{w}$

b) *Cost minimization for the joint firm:*

Problem: Min (L_a, L_b): $w\,(L_a + L_b)$ s.t. $Q = L_a^{1/2} + 2\,L_b^{1/2}$

Then, $£ = w\,(L_a + L_b) = \lambda\,(Q - L_a^{1/2} + 2\,L_b^{1/2}\,)$

FOC :

$\partial£ / \partial L_a = w - \lambda\,1/2\,L_a^{-1/2} = 0$
$\partial£ / \partial L_b = w - \lambda\,L_b^{-1/2} = 0$
$\partial£ / \partial\,\lambda = Q - L_a^{1/2} + 2\,L_b^{1/2} = 0$

Solving for w (1) and (2) and equalizing them:

$1/2\ L_a^{-1/2} = L_b^{-1/2}$ then $2\ L_a^{1/2} = L_b^{1/2}$

Then $Q = L_a^{1/2} + 2\ L_b^{1/2} = L_a^{1/2} + 4\ L_a^{1/2} = 5\ L_a^{1/2}$

Solving this for L_a we get: $L_a^* = Q^2 / 25$ and then $L_b^* = 4\ L_a = 4\ Q^2 / 25$

Then $C^*(Q) = w(\frac{Q^2}{25} + 4\frac{Q^2}{25}) = \frac{wQ^2}{5}$

Profit maximization for the joint firm:

Max (Q): p Q – C(Q)

FOC:

$\partial\pi / \partial Q = p - MC = 0$ and MC = $\frac{2wQ}{5}$ so

$$Q^* = \frac{5p}{2w}$$

c) *Output supply*

Is the function $Q^* = Q(p)$ that we found in point (b) so

Supply = $Q^* = \frac{5p}{2w}$

From cost minimization we found that:

$2\ L_a^{1/2} = L_b^{1/2}$ so $2\ Q_a = 1/2\ Q_b$ then $4Q_a = Q_b$

Remember, $MC_a = 2w\ Q_a$
$MC_b = w/2\ Q_b = 2w\ Q_a$

So we see that marginal cost must be equal at both facilities.